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Remark on Periodic Solutions of Non Linear Oscillators

A. Raouf Chouikha

University of Paris-Nord
Institut Galilee
LAGA,CNRS UMR 7539
Villetaneuse F-93430
Chouikha@math.univ-paris13.fr

Abstract

We contribute to the method of trigonometric series for solving differential equations of certain non linear oscillators.
Key Words: series power solution, trigonometric series.[1]

1 Introduction

The non linear nonharmonic motion of an oscillator may be given by the following differential equation

$$u'' + \omega^2 u = -\beta u^2 \tag{1}$$

β and ω being constants, with initial conditions

$$u(0) = a_0, \qquad u'(0) = 0 \tag{2}$$

To solve this problem A. Shidfar and A. Sadeghi [1], have given two series solutions. They describe a general approach in which the differential equation, rather than the solutions series , is majorized.
Notice that if we write $a_0 = -\frac{\omega^2}{\beta}$ then

$$u(t) \equiv -\frac{\omega^2}{\beta} \tag{3}$$

is a trivial solution of (1) and (2).
They gave a series solutions of (1) and (2), which includes (3) as a special

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case.

By writing

$$u = v - \frac{\omega^2}{2\beta}$$

the problem becomes

$$v'' + \beta v^2 = \frac{\omega^4}{4\beta} \tag{4}$$

under the initial conditions

$$\begin{cases} v(0) = a_0 + \frac{\omega^2}{2\beta} \\ v'(0) = 0 \end{cases} \tag{5}$$

The method of [1] consists to solve equations (4) and (5) in the form

$$v(t) = c_0 + c_1 \sin \omega t + c_2 \sin^2 \omega t + c_3 \sin^3 \omega t + ... \tag{6}$$

where $c_i, i = 0, 1, 2, ...$ are coefficients to be determined by the substitution of (6) in (4).
In fact, $\omega = \frac{\pi}{T}$ where T is the period of the solution, which can be expressed in terms of the Weierstrass function $\wp(z, 2T, 2T')$.
So, we find that

$$2\omega^2 c_2 + \beta c_0^2 = \frac{\omega^2}{4\beta}.$$

For $n \geq 1$, the recursion formula for these coefficients are

$$(n+1)(n+2)c_{n+2} = n^2 c_n - \frac{\beta}{\omega^2} \sum_{r=0}^{n} c_r c_{n-r}. \tag{7}$$

Equations (5) and (6) imply that $c_1 = 0$. Relations (7) yields

$$c_3 = 0, \quad c_5 = 0, ...$$

The even order coefficients simply are

$$c_0 = a_0 + \frac{\omega^2}{2\beta},$$

$$c_2 = -\frac{a_0}{2\omega^2}(\omega^2 + a_0\beta),$$

$$c_4 = -\frac{\beta}{6\omega^2} a_0(\omega^2 + a_0\beta)(\frac{3}{4} - \frac{a_0\beta}{2\omega^2}),$$

$$c_6 = -\frac{\beta}{180\omega^2} a_0(\omega^2 + a_0\beta)(\frac{3}{4} - \frac{a_0\beta}{2\omega^2})(15 - \frac{2a_0\beta}{\omega^2}) - \frac{\beta a_0^2}{120\omega^6}(\omega^2 + a_0\beta)^2,$$

etc.
The coefficient c_0 follows from the condition (5). The solution for the equations (4) and (5) can now be written as

$$u(t) = a_0 - \frac{a_0}{2\omega^2}(\omega^2 + a_0\beta)\sin^2 \omega t + ... \tag{8}$$

Relations of the coefficients and further induction show that $c_{2i},\ i = 1, 2, ...$ all vanish for $a_0 = \frac{\omega^2}{\beta}$. So, the trivial solution (3) is included in (6) as a special case.

2 Convergence of the solutions

We now show the convergence of these series. In [1] one proved the following

Lemma 1 *The serie (6) solution of Equation (4)-(5) is absolutely convergent for all t.*

Proof We firstly note that if $c_0 > 0,\ c_2 > 0$ and $\beta < 0,$ then all coefficients c_n in the serie expansion (6) are positive. Indeed, we may write

$$\sum_{n\geq 0}((n+1)(n+2)c_{n+2} = \sum_{n\geq 0} n^2 c_n - \frac{\beta}{\omega^2}(\sum_{n\geq 0} c_n)^2 + \frac{\beta}{\omega^2}c_0^2,$$

or

$$-\beta(\sum_{n\geq 0} c_n)^2 + \omega^2 \sum_{n\geq 0} nc_n = -\beta c_0^2 - 2\omega^2 c_2. \tag{9}$$

Since the right hand side of (9) is finite and c_i are positive, the series $\sum_{n\geq 0} c_n$ converges. Following [1], if we put

$$c'_0 =\mid c_0 \mid, \quad c'_1 = 0, \quad c'_2 =\mid c_2 \mid$$

and for $n \geq 2$

$$c'_{n+2} = \frac{n^2}{(n+1)(n+2)}c'_n + \frac{\mid \beta \mid}{\omega^2(n+1)(n+2)}\sum_{r=0}^{n} c'_r c'_{n-r},$$

then the series $\sum_{n\geq 0} c'_n$ converges. Since $\mid c_n \mid \geq c'_n,$ it follows that the solution series (6) is absolutely convergent, and hence the series expansion solution of (1)-(2) converges for all t.

We notice that we may deduce Lemma 1 a previous result concerning Equation (4).
We have shown that the coefficients verify a more general properties. Indeed, we have [4]

Lemma 2 *For any positive number* ϵ *small enough (but* $\epsilon \neq 0$*), there exists a positive constant* k *verifying*

$$k < \frac{\beta}{\omega^2}\frac{3}{4}\epsilon 4^{\epsilon-\frac{1}{2}}$$

such that the coefficients c_n *of the series expansion (6) solution of the differential equation (4)-(5) satisfy the inequality*

$$\mid c_n \mid < \frac{k}{n^{\frac{3}{2}-\epsilon}}. \tag{10}$$

Proof We first notice that Lemma 2 gives an optimal result, because our method do not run for $\epsilon = 0$.
The coefficients c_n of the power series solution, satisfy the recursion formula (7). We shall prove there exist two positive constants $k > 0$, and $\alpha > 1$, such that the following inequality holds

$$\mid c_n \mid < \frac{k}{n^\alpha}$$

for any integer $n \geq 1$. Suppose for any $n \leq p$, we get $\mid c_n \mid < \frac{k}{n^\alpha}$. In particular, it implies that

$$\sum_{0<r<p} c_r c_{p-r} < \sum_{0<r<p} \frac{k^2}{r^\alpha (r-p)^\alpha} \leq \frac{k^2}{(p-1)^{\alpha-1}}.$$

Equality (7) gives

$$c_{p+2} = \frac{p^2 - 2\frac{\beta}{\omega^2}c_0}{(p+1)(p+2)}c_p - \frac{\beta}{\omega^2(p+1)(p+2)}\sum_{r=1}^{p-1} c_r c_{p-r}.$$

Thus, if we prove the following inequality

$$\frac{p^2 - 2\frac{\beta}{\omega^2}c_0}{(p+1)(p+2)}\frac{k}{p^\alpha} + \frac{\beta}{\omega^2(p+1)(p+2)}\frac{k^2}{(p-1)^{\alpha-1}} \leq \frac{k}{(p+2)^\alpha} \tag{11}$$

so

$$\mid c_{p+2} \mid < \frac{k}{(p+2)^\alpha} \tag{12}$$

Notice that (11) implies that

$$\frac{p^2 - 2\frac{\beta}{\omega^2}c_0}{(p+1)(p+2)}\frac{k}{p^\alpha} \leq \frac{k}{(p+2)^\alpha}.$$

Thus, a necessary condition to (11) holds is : $\alpha \leq \frac{3}{2}$. Inequality (11) is equivalent to

$$k < \frac{\beta}{\omega^2} p f(p) g(p) \tag{13}$$

where

$$f(p) = \frac{p+1}{p} (\frac{p-1}{p+2})^{\alpha - 1}$$

$$g(p) = 1 - \frac{(p^2 - \frac{\beta}{\omega^2} c_0)(p+2)^{\alpha-1}}{(p+1)p^{\alpha}}$$

By using MAPLE, we are able to prove that $f(p)$ is an increasing positive function in p . Moreover, for any $p \geq 1$, $f(p)$ is minorated

$$f(p) \geq (\frac{3}{2}) 4^{1-\alpha}.$$

The function $g(p)$ is such that

$$pg(p) = p - \frac{(p^2 - \frac{\beta}{\omega^2} c_0)(\frac{p+2}{p})^{\alpha-1}}{(p+1)}$$

is a strictly decreasing and bounded function .
More exactly, we may calculate the lower bound

$$g(p) > \frac{(3 - 2\alpha)}{p}.$$

Thus, if $(3 - 2\alpha) = \epsilon > 0$, it suffices to choice

$$k \leq (\frac{3}{2}) 4^{1-\alpha} (3 - 2\alpha)$$

to inequality (13) holds.

Remark for the case $\epsilon = 0$:
Notice that the choice of k depends on α value.
For $\alpha = \frac{3}{2}$, we then prove by MAPLE that the function

$$pg(p) = p - \frac{(p^2 - \frac{3}{2\omega^2} c_0)(\frac{p+2}{p})^{\frac{1}{2}}}{(p+1)}$$

is positive and strictly decreasing to 0. While $p^2 g(p)$ is a bounded function. Moreover, it appears that $pf(p)g(p)$ is a decreasing function which tends to 0 when p tends to infinity. Thus, our method falls since it do not permit to determine a non negative constant k.

Following [1], it is interesting to write the power series solution for the system (4)-(5),

$$v(x) = \sum_{n=0}^{\infty} b_n x^n. \tag{14}$$

We find again that

$$b_{2p+1} = 0 \qquad p = 0, 1, 2, ...,$$

while

$$b_0 = a_0 + \frac{\omega^2}{2\beta},$$

$$2b_2 + \beta b_0^2 = \frac{\omega^4}{4\beta},$$

$$(n+2)(n+1)b_{n+2} = -\beta \sum_{r=0}^{r=n} b_r b_{n-r},$$

where n is even and non zero.
The coefficients $b_{2p}, p = 0, 1, 2, ...,$ again vanish for $a_0 = -\frac{\omega^2}{\beta}$. We may verify that the solutions (6) and (14) are identical.
The latter method permits to compare approximate solutions of the anharmonic motion of the oscillator.

3 Another differential equation

We now examine the following differential equation

$$u'' + \omega^2 u = -\beta u^3 \tag{15}$$

β and ω being constants, with initial conditions

$$u(0) = a_0, \qquad u'(0) = 0. \tag{16}$$

We put $v = \frac{u}{a_0}$ and $t = \omega x$. We then obtain from (15) and (16)

$$\frac{d^2v}{dt^2} + v + \beta v^3 = 0, \quad v(0) = 1, \quad v'(0) = 0 \tag{17}$$

where $\beta = \frac{\beta a_0^2}{\omega}$.
A. Shidfar and A. Sadeghi [2] solved (17) by series method in Sinus power

$$v(t) = c_0 + c_1 \sin \omega t + c_2 \sin^2 \omega t + c_3 \sin^3 \omega t + ... \tag{18}$$

Here, $\omega = \frac{\pi}{T}$ where T is the period of the solution, which can be expressed in terms of the Jacobi function $sn(z, 2T, 2T')$.
So,

$$c_0 = a_0.$$

For $n \geq 1$, we get the recursion formula

$$(n+1)(n+2)c_{n+2} = n^2 c_n - \frac{\beta}{\omega^2} \sum_{r=0}^{n} \sum_{m=0}^{n-r} c_m c_r c_{n-m-r}. \tag{19}$$

Under some conditions, they proved estimates of the coefficients

$$| c_n | \leq R^n$$

where $\frac{1}{R}$ is a radius of convergence.
In fact, we may prove an analog of Lemma 2 for this equation. Indeed, we have

$$\sum_{r=0}^{n} \sum_{m=0}^{n-r} c_m c_r c_{n-m-r} = 2c_0^2 c_n + 2c_0 c_1 c_{n-1} + c_0 \sum_{m=0}^{n} c_n c_{n-m} + \sum_{r=2}^{n-2} \sum_{m=0}^{n-r} c_m c_r c_{n-m-r}.$$

Then,

$$\sum_{r=2}^{n-2} \sum_{m=0}^{n-r} c_m c_r c_{n-m-r} = \sum_{r=2}^{n-2} c_r [2c_0 c_{n-r} + \sum_{m=1}^{n-r-1} c_m c_{n-m-r}]$$

$$< \sum_{r=2}^{n-2} | c_r | [2 | c_0 c_{n-r} | + \frac{k^2}{(n-r-1)^{\alpha-1}}]$$

$$< 2c_0 k^2 \sum_{r=2}^{n-2} \frac{1}{r^\alpha (n-r)^\alpha} + k^3 \sum_{r=2}^{n-2} \frac{1}{r^\alpha (n-r-1)^{\alpha-1}} < \frac{2c_0 k^2}{(n-1)^{\alpha-1}} + \frac{k^3}{(n-2)^{\alpha-1}}$$

Finally,

$$| c_{p+2} | < \frac{k}{(p+2)^\alpha}$$

as soon as the non negative constant k satisfies the inequality

$$\frac{k}{n^{\alpha-2}} + \frac{2c_0 k^2}{n^\alpha} + \frac{2c_0 c_1 k}{(n-1)^\alpha} + \frac{3c_0 k^2}{(n-2)^{\alpha-1}} + \frac{k^3}{(n-2)^{\alpha-1}} < \frac{(n+1)k}{(n+2)^{\alpha-1}},$$

So,

$$\frac{1}{n^{\alpha-2}} + \frac{k^2 + 3c_0 k}{(n-2)^{\alpha-1}} + \frac{2c_0 + 2c_0 c_1}{(n-1)^\alpha} < \frac{(n+1)}{(n+2)^{\alpha-1}}.$$

By using MAPLE, we verify it is possible to find a such constant. Moreover, we find again the necessary condition : $\alpha \leq \frac{3}{2}$, since we get $\frac{1}{n^{\alpha-2}} < \frac{(n+1)}{(n+2)^{\alpha-1}}$.

General remarks : It is wellknown from the theory of elliptic functions that solutions of equations (4) and (17) are related. This allows to express the series expansion of the solution of (17) from a series expansion of a solution of (4) and conversely. Indeed, one has

$$\wp(z) = C - \frac{\delta^2}{sn^2(\delta z)},$$

where $\wp(z)$ is the elliptic Weierstrass function and $sn(u)$ is the Jacobi fonction , δ is a constant , only dependent on the initial parameters.
Notice that series expansion of the $sn(u)$ in sinus power was given in a previous paper (see Proposition (2.1) in [3]).

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